Internship report

Introduction to Harmonic Analysis and Maximal Functions

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1. Introduction and Notation

In this paper we first extend the definition of the Fourier transform, and then state some important and fundamental results, especially about some convergence results. Unless otherwise stated, we will use the Lebesgue measure in $\mathbf{R}^n$.

1.1. Extension of the Fourier transform

For a function $f \in \mathrm{L}^1(\mathbf{R}^n)$ we define its Fourier transform by

$$\hat{f}(\xi) := \int_{\mathbf{R}^n} f(x) e^{-2i\pi x\cdot\xi} \mathrm{d}x$$

where $\xi \in \widehat{\mathbf{R}}^n$. The map

$$\mathcal{F} : \left| \begin{array}{ccc} \mathrm{L}^1(\mathbf{R}^n) & \longmapsto & \mathrm{L}^\infty(\widehat{\mathbf{R}}^n) \\ f & \longrightarrow & \hat{f} \end{array} \right.$$

is then a bounded linear operator. All the basic results about the Fourier transform can be found in [1]. The notation $\widehat{\mathbf{R}}$ is just another notation for $\mathbf{R}$, but it's useful to show that we are working the frequency space. However let's recall some important results.

Theorem 1.1 (Plancherel). *$\mathcal{F}$ can be extended as a bijective operator $\mathrm{L}^2(\mathbf{R}^n) \longrightarrow \mathrm{L}^2(\widehat{\mathbf{R}}^n)$. Furthermore if $f \in \mathrm{L}^2(\mathbf{R}^n)$, then*

$$\|f\|_{\mathrm{L}^2(\mathbf{R}^n)} = \|\hat{f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)}.$$

We still denote $\mathcal{F}$ this extended operator. Since $\mathcal{F}$ is well defined on $\mathrm{L}^1(\mathbf{R}^n)$ and on $\mathrm{L}^2(\mathbf{R}^n)$, we can define it by interpolation on $\mathrm{L}^p(\mathbf{R}^n)$ for $1 < p < 2$.

Theorem 1.2 (Riesz-Thorin Interpolation). *Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$, and for $0 < \theta < 1$ define p and q by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T is a linear operator from $\mathrm{L}^{p_0}(\mathbf{R}^n) + \mathrm{L}^{p_1}(\mathbf{R}^n)$ to $\mathrm{L}^{q_0}(\mathbf{R}^n) + \mathrm{L}^{q_1}(\mathbf{R}^n)$ such that

$$\|Tf\|_{\mathrm{L}^{q_0}(\mathbf{R}^n)} \leqslant C_0 \|f\|_{\mathrm{L}^{p_0}(\mathbf{R}^n)}, \quad f \in \mathrm{L}^{p_0}(\mathbf{R}^n)$$

and

$$\|Tf\|_{\mathrm{L}^{q_1}(\mathbf{R}^n)} \leqslant C_1 \|f\|_{\mathrm{L}^{p_1}(\mathbf{R}^n)}, \quad f \in \mathrm{L}^{p_1}(\mathbf{R}^n)$$

Where C_0, C_1 are some real constant. Then T is defined on $\mathrm{L}^p(\mathbf{R}^n)$ and

$$\|Tf\|_{\mathrm{L}^q(\mathbf{R}^n)} \leqslant C_0^{1-\theta} C_1^\theta \|f\|_{\mathrm{L}^{p_0}(\mathbf{R}^n)}, \quad f \in \mathrm{L}^p(\mathbf{R}^n).$$

The proof can be found in [9]. We apply this theorem with the Fourier transform $\mathcal{F}$ and with the inequalities

$$\|\hat{f}\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} \leqslant \|f\|_{\mathrm{L}^1(\mathbf{R}^n)} \quad \text{and} \quad \|\hat{f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)} = \|f\|_{\mathrm{L}^2(\mathbf{R}^n)}.$$

Corollary 1.3 (Hausdorff-Young inequality). *Let $1 \leqslant p \leqslant 2$. If $f \in \mathrm{L}^p(\mathbf{R}^n)$ then $\hat{f} \in \mathrm{L}^{p'}(\widehat{\mathbf{R}}^n)$ and*

$$\|\hat{f}\|_{\mathrm{L}^{p'}(\widehat{\mathbf{R}}^n)} \leqslant \|f\|_{\mathrm{L}^p(\mathbf{R}^n)}.$$

1.2. Convergence result

One of the main problem in harmonic analysis is to make sense of the inversion formula

$$f(x) = \int_{\widehat{\mathbf{R}}^n} \hat{f}(\xi) e^{2i\pi x\cdot\xi} \mathrm{d}\xi \tag{1.1}$$

for $x \in \mathbf{R}^n$. If $f \in \mathrm{L}^1(\mathbf{R}^n)$ and $\hat{f} \in \mathrm{L}^1(\widehat{\mathbf{R}}^n)$, then (1.1) holds for almost every $x \in \mathbf{R}^n$. But we saw that if $f \in \mathrm{L}^p(\mathbf{R}^n)$ for $1 < p < 2$, then $\hat{f} \in \mathrm{L}^{p'}(\widehat{\mathbf{R}}^n)$ and so there is nothing to ensure the integrability of $\hat{f}$. However the function $\hat{f}$ is at least locally integrable. So we can define the **partial Fourier integrals**,

$$S_R f : x \in \mathbf{R}^n \longmapsto S_R f(x) := \int_{[-R,R]^n} \hat{f}(\xi) e^{2i\pi x\cdot\xi} \mathrm{d}\xi$$

for $f \in \mathrm{L}^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant 2$, and $R > 0$. So one of the main problem in harmonic analysis is to study the convergence of $S_R f$ as $R \to \infty$. We have two results of convergence for the Fourier integrals.

Theorem 1.4 (M. Riesz)**.** *Let $1 < p \leqslant 2$. If $f \in \mathrm{L}^p(\mathbf{R}^n)$ then*

$$\|S_R f - f\|_{\mathrm{L}^p(\mathbf{R}^n)} \xrightarrow[R\to\infty]{} 0.$$

Theorem 1.5 (Carleson-Hunt)**.** *Let $1 < p \leqslant 2$. If $f \in \mathrm{L}^p(\mathbf{R}^n)$, then*

$$S_R f \xrightarrow[R\to\infty]{} f, \quad \textit{almost everywhere.}$$

Proving these results are quite complicated, and we are not going to prove it here. However, we will introduce in this paper some basic tools and some important theories in harmonic analysis.

2. The Hardy-Littlewood Maximal Function

We use in this paper the notation of the average $\fint$, ie

$$\fint_A f\mathrm{d}\mu := \frac{1}{\mu(A)}\int_A f\mathrm{d}\mu.$$

B_r designates the ball centered at the origin and with a radius r. We note $B(x,r)$ for the ball of radius r centered on x.

2.1. Introduction and definition

Definition 2.1. For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$, we define the **average operator** associated to f as

$$A_r f(x) := \fint_{B_r} f(x-y)\mathrm{d}y,$$

where $r > 0$ and $x \in \mathbf{R}^n$.

We can interpret the average operator as a convolution, for $r > 0$ and $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$:

$$A_r f = \frac{1}{|B_r|}\chi_{B_r} * f.$$

A simple consequence of Young's inequality shows that for all $f \in \mathrm{L}^p(\mathbf{R}^n)$, if $1 \leqslant p \leqslant \infty$,

$$\|A_r f\|_{\mathrm{L}^p(\mathbf{R}^n)} \leqslant \|f\|_{\mathrm{L}^p(\mathbf{R}^n)}. \tag{2.1}$$

Definition 2.2. For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$, we define the associated **Hardy-Littlwood** maximal function as

$$M_{\mathrm{HL}} f(x) := \sup_{r>0} A_r|f|(x),$$

where $x \in \mathbf{R}^n$.

A priori, this quantity could be equal to ∞. Sometimes it could be useful to consider this maximal function on some cubes centered on $x \in \mathbf{R}^n$, and not necessarily on balls. That's why we introduce the equivalent function

$$M'_{\mathrm{HL}} f(x) := \sup_{r>0}\fint_{Q_r} |f(x-y)|\mathrm{d}y$$

where $Q_r := [-r,r]^n$. We can assume that these functions are equivalent in a certain way because there are some constants c_n, C_n such that

$$c_n M'_{\mathrm{HL}} f(x) \leqslant M_{\mathrm{HL}} f(x) \leqslant C_n M'_{\mathrm{HL}} f(x), \tag{2.2}$$

for all $x \in \mathbf{R}^n$. In fact, any cubes contain a smaller ball inversely.

Example 2.3. Let $f := \chi_B$, where B denotes the unit ball centered at the origin in $\mathbf{R}^n$. The notation $\lesssim$ means $\leqslant C$ where C is a constant. In this case

$$M_{\mathrm{HL}} f(x) \geqslant \frac{|B \cap B(x,2|x|)|}{|B(x,2|x|)|} \gtrsim (1+|x|)^{-n}$$

for all $r > 0$ and $x \in \mathbf{R}^n$.

This example show that, unlike the average operator A_r (see (2.1)), the Hardy-Littlewood maximal function is not bounded in $\mathrm{L}^1(\mathbf{R}^n)$. Indeed

$$\int_{\mathbf{R}^n} (1+|x|)^{-n} \mathrm{d}x = \infty.$$

However, we will prove that M_{HL} function is bounded on $\mathrm{L}^p(\mathbf{R}^n)$ for $1 < p \leqslant \infty$, and also admits some "weak-bound" on $\mathrm{L}^1(\mathbf{R}^n)$. The following result shows the important of the Hardy-Littlewood maximal function, especially to study some approximations of the identity.

Proposition 2.4. *Let $\varphi : \mathbf{R} \longrightarrow \mathbf{R}$ be a function which is positive, radial, decreasing, and integrable. We note for $t > 0$*

$$\varphi_t = t^{-1}\varphi(t^{-1}\cdot).$$

Then

$$\sup_{t>0} |\varphi_t * f(x)| \leqslant \|\varphi\|_{\mathrm{L}^1(\mathbf{R})} M_{\mathrm{HL}} f(x)$$

for all $x \in \mathbf{R}$ and for all $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R})$.

Like we said, this theorem can be very interesting if $\{\varphi_t\}_t$ is an approximation of the identity. So it would be useful to obtain some bounds for M_{HL}.

2.2. **The Marcinkiewicz interpolation theorem**

Definition 2.5. Let (X,μ), (Y,ν) be two measure spaces and T an operator from $\mathrm{L}^p(X,\mu)$ into the space of measurable complex-valued functions $Y \to \mathbf{C}$, with $1 \leqslant p \leqslant \infty$. We say that

- T is **weak** (p,q), for $q < \infty$, if

  $$\nu(\{|Tf| > \lambda\}) \lesssim \left(\frac{\|f\|_{\mathrm{L}^p(X,\mu)}}{\lambda}\right)^q$$

  for all $\lambda > 0$ and for all $f \in \mathrm{L}^p(X,\mu)$.
- T is weak (p,∞) if T is a bounded operator from $\mathrm{L}^p(X,\mu)$ to $\mathrm{L}^\infty(Y,\nu)$.
- T is **strong** (p,q) if T is a bounded operator from $\mathrm{L}^p(X,\mu)$ to $\mathrm{L}^q(Y,\nu)$, ie

  $$\|Tf\|_{\mathrm{L}^q(Y,\nu)} \lesssim \|f\|_{\mathrm{L}^p(X,\mu)}$$

  for all $f \in \mathrm{L}^p(X,\mu)$.

Proposition 2.6. *A strong (p,q) operator is weak (p,q).*

Proof. Let's use the notation of the previous definition. Let $f \in \mathrm{L}^p(X,\nu)$ et $\lambda > 0$. We assume that $q < \infty$ (otherwise the result is obvious). Then

$$\nu(\{|Tf| > \lambda\}) = \int_{\{|Tf|>\lambda\}} \mathrm{d}\nu \leqslant \int_{\{|Tf|>\lambda\}} \left|\frac{Tf(y)}{\lambda}\right|^q \mathrm{d}\nu(y) \leqslant \frac{\|Tf\|^q_{\mathrm{L}^q(\mathbf{R}^n)}}{\lambda^q}.$$

Since T is strong (p,q), we have

$$\nu(\{|Tf| > \lambda\}) \lesssim \left(\frac{\|f\|_{\mathrm{L}^p(\mathbf{R}^n)}}{\lambda}\right)^q,$$

which concludes the proof. □

Definition 2.7. An operator T from a vector space of measurable functions $\mathcal{F}$ into another space of measurable functions is said to be **sublinear** if

$$|T(f_0 + f_1)(\cdot)| \leqslant |Tf_0(\cdot)| + |Tf_1(\cdot)|, \quad \forall f_0, f_1 \in \mathcal{F},$$
$$|T(\lambda f)(\cdot)| = |\lambda||Tf|, \quad \forall f \in \mathcal{F}, \forall \lambda \in \mathbf{C}.$$

Example 2.8. The Hardy-Littlwood maximal function is a sublinear operator.

Theorem 2.9 (Marcinkiewicz Interpolation)**.** *Let (X,μ), (Y,ν) be two measure spaces and let T be a sublinear operator from $\mathrm{L}^{p_0}(X,\mu) + \mathrm{L}^{p_1}(X,\mu)$, with $1 \leqslant p_0 < p_1 \leqslant \infty$. We assume that T is weak (p_0,p_0) and weak (p_1,p_1). Then T is strong (p,p) for all $p_0 < p < p_1$.*

We already mentioned an interpolation theorem with the theorem of Riesz-Thorin for the linear operator. We introduce a new one, which applies to the sublinear operator. So with this theorem, it can be enough to prove some weak bounds on the operators, and it is often easier, hence the interest of this theorem. The following lemma will help us to prove it.

Lemma 2.10. *Let (X,μ) be a measured space and $f \in \mathrm{L}^p(X,\mu)$. Then*

$$\|f\|^p_{\mathrm{L}^p(X,\mu)} = p\int_0^\infty \lambda^{p-1}\mu(\{|f| > \lambda\})\mathrm{d}\lambda.$$

Proof of the lemma. We notice that for all $x \in X$,

$$|f(x)|^p = p\int_0^{|f(x)|} \lambda^{p-1}\mathrm{d}\lambda.$$

Then

$$\|f\|^p_{\mathrm{L}^p(\mathbf{R}^n)} = \int_X \int_0^{|f(x)|} p\lambda^{p-1}\mathrm{d}\lambda\mathrm{d}\mu(x),$$

by Fubini-Tonelli's theorem we can switch the integrals

$$\begin{aligned}\|f\|^p_{\mathrm{L}^p(\mathbf{R}^n)} &= \int_0^\infty \int_{\{x\in X:|f(x)|>\lambda\}} p\lambda^{p-1}\mathrm{d}\lambda\mathrm{d}\mu(x)\\ &= p\int_0^\infty \lambda^{p-1}\mu(\{|f| > \lambda\})\mathrm{d}\lambda.\end{aligned}$$

□

Let's move on the proof of Marcinkiewicz's theorem.

Proof of the theorem. Let $f \in \mathrm{L}^p(X,\mu)$, for $p_0 < p < p_1$, and let $\lambda > 0$. We give us a constant c, which the value will be fixed later. We decompose f as $f_0 + f_1$, where

$$f_0 = f\chi_{\{|f|>c\lambda\}}$$

and

$$f_1 = f\chi_{\{|f|\leqslant c\lambda\}},$$

(we decompose f as a sum of her large values and her small values). Since $p_0 < p$, we can write

$$\int_X |f_0|^{p_0}\mathrm{d}\mu = \int_X |f|^p|f|^{p_0-p}\chi_{\{|f|>c\lambda\}}\mathrm{d}\mu \leqslant (c\lambda)^{p-p_0}\|f\|^p_{\mathrm{L}^p(X,\mu)} < \infty.$$

By an analogous proof, we can conclude that $f_0 \in \mathrm{L}^{p_0}(X,\mu)$ and $f_1 \in \mathrm{L}^{p_1}(X,\mu)$. Since T is a sublinear operator from $\mathrm{L}^{p_0}(X,\mu) + \mathrm{L}^{p_1}(X,\mu)$, we have for all $x \in X$

$$|Tf(x)| \leqslant |Tf_0(x)| + |Tf_1(x)|.$$

Comparating $|Tf_0(x)|$ and $|Tf_1(x)|$, we obtain

$$\mu(\{|Tf| > \lambda\}) \leqslant \mu(\{|Tf_0| > \lambda/2\}) + \mu(\{|Tf_1| > \lambda/2\}).$$

We consider two cases.

- If $p_1 = \infty$. By the weak (p_0, p_0) and (p_1, p_1) inequalities, there exist two constants A_0, A_1 such that

$$\mu(\{|Tf_0| > \lambda/2\}) \leqslant \left(\frac{2A_0}{\lambda}\|f_0\|_{\mathrm{L}^{p_0}(X,\mu)}\right)^{p_0} \tag{2.3}$$

and

$$\|Tf_1\|_{\mathrm{L}^\infty(X,\mu)} \leqslant A_1\|f_1\|_{\mathrm{L}^\infty(X,\mu)}.$$

So by choosing $c = 1/2A_1$, we have

$$\mu(\{|Tf_1| > \lambda/2\}) \leqslant \mu(\{A_1|f_1| > \lambda/2\}) = 0.$$

Hence, by using the lemma 2.10 and the weak inequality (2.3),

$$\begin{aligned}\|Tf\|^p_{\mathrm{L}^p(X,\mu)} &\leqslant p\int_0^\infty \lambda^{p-1}\mu(\{|Tf_0| > \lambda/2\})\mathrm{d}\lambda \\ &\leqslant p(2A_0)^{p_0}\int_0^\infty \lambda^{p-p_0-1}\int_{\{|f|>c\lambda\}} |f|^{p_0}\mathrm{d}\mu\mathrm{d}\lambda.\end{aligned}$$

Fubini-Tonelli's theorem allows us to conclude,

$$\begin{aligned}\|Tf\|^p_{\mathrm{L}^p(X,\mu)} &\leqslant p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{|f(x)|/c} \lambda^{p-p_0-1}\mathrm{d}\lambda\mathrm{d}\mu(x) \\ &= \frac{p2^p}{p-p_0}A_0^{p_0}A_1^{p-p_0}\|f\|^p_{\mathrm{L}^p(X,\mu)}.\end{aligned}$$

- If $p_1 < \infty$. This time we have

$$\mu(\{|Tf_i| > \lambda/2\}) \leqslant \left(\frac{2A_i}{\lambda}\|f_i\|_{\mathrm{L}^{p_i}(X,\mu)}\right)^{p_i}$$

for $i \in \{0, 1\}$. From this we get, after a similar calculation,

$$\|Tf\|^p_{\mathrm{L}^p(X,\mu)} \leqslant p\left(\frac{(2A_0)^{p_0}}{p-p_0}c^{p_0-p} + \frac{(2A_1)^{p_1}}{p_1-p}c^{p_1-p}\right)\|f\|_{\mathrm{L}^p(X,\mu)}.$$

In both cases, we can assert that T is strong (p, p). □

2.3. **Dyadic maximal operator**

In harmonic analysis, it is often easier to work on discrete objects instead of continous object. This one of the reason for us to introduce a decomposition of $\mathbf{R}^n$ into some cubes of different length : the dyadic cubes.

2.3.1. **Dyadic cubes**

Definition 2.11. A **dyadic cube** in $\mathbf{R}^n$ is a subset of the form

$$Q = \prod_{i=1}^{n} [k_j 2^r, (k_j+1)2^r[$$

where $h_1, \ldots, k_n \in \mathbf{Z}$ and $r \in \mathbf{Z}$. For a such cube Q, we define his length by $l(Q) := 2^r$. We denote by $\mathcal{D}^n$ the collection of dyadic cubes in $\mathbf{R}^n$, and

$$\mathcal{D}^n_r := \{Q \in \mathcal{D}^n : l(Q) = 2^r\}.$$

From the definition we can easily deduce the following propositions.

Proposition 2.12. *(i) For all $x \in \mathbf{R}^n$ and for all $r \in \mathbf{Z}$, there exists a unique cube $Q \in \mathcal{D}^n_r$ such that $x \in Q$.*

(ii) Two dyadic cubes are either disjoint or one contains the other.

(iii) If $r, s \in \mathbf{Z}$ with $r < s$, then each cube $Q \in \mathcal{D}^n_r$ is contained in a unique cube in $\mathcal{D}^n_s$.

Let's represent some of these cubes in the portion of plane $[-1,1] \times [-1,1]$ of $\mathbf{R}^2$. For a given x, it's clear that for each fixed length, x lies in a unique dyadic cube.

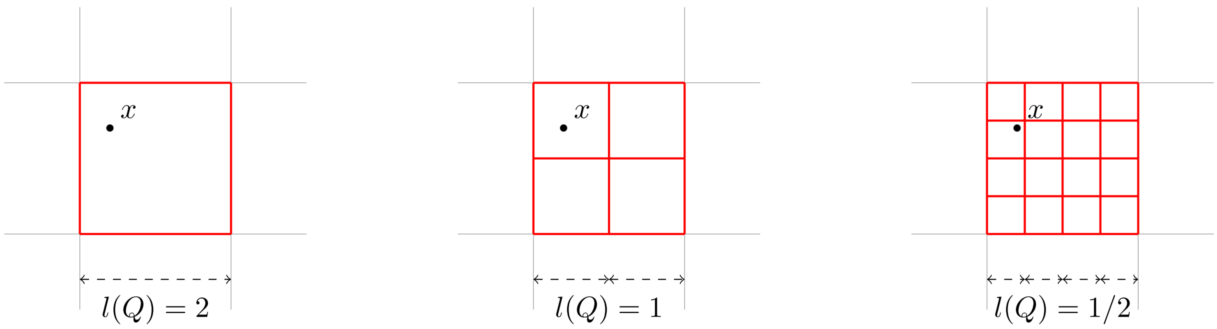

Figure 1: $[-1,1]^2 \cap \mathcal{D}^2_r, r \in \{-1, 0, 1\}$.

2.3.2. **Dyadic maximal function**

Definition 2.13. For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $k \in \mathbf{Z}$, we note

$$E_k f(x) := \sum_{Q \in \mathcal{D}^n_k} \left(\fint_Q f \right) \mathbf{1}_Q(x)$$

où $x \in \mathbf{R}^n$. We define the **dyadic maximal function** associated to f as

$$M_d f(x) := \sup_{k \in \mathbf{Z}} E_k |f|(x).$$

The operators $\{E_k : k \in \mathbf{Z}\}$ satisfy the fundamental identity : if Ω is the union of cubes in $\mathcal{D}_k^n$ then

$$\int_\Omega E_k f = \int_\Omega f \tag{2.4}$$

for all $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$. Hence, $\{E_k : k \in \mathbf{Z}\}$ can be seen as a **discrete approximation of the identity**.

Theorem 2.14. *The dyadic maximal function is weak* $(1,1)$.

Proof. Let $f \in \mathrm{L}^1(\mathbf{R}^n)$. Because of the definition of M_d, we may assume that f is non-negative. For $k \in \mathbf{Z}$ we define the set Ω_k as

$$\Omega_k := \{x \in \mathbf{R}^n : E_k f(x) > \lambda \text{ and } E_j f(x) \leqslant \lambda \text{ if } j > k\}.$$

Then we can see that

$$\{x \in \mathbf{R}^n : M_d f(x) > \lambda\} = \bigsqcup_{k \in \mathbf{Z}} \Omega_k.$$

In fact, if $x \in \mathbf{R}^n$ is such that $M_d f(x) > \lambda$, then the sets $\{k \in \mathbf{Z} : E_k f(x) > \lambda\}$ are not empty. Since $E_k f(x) \to 0$ as $k \to \infty$ (because $f \in \mathrm{L}^1(\mathbf{R}^n)$), we are allowed to choose k as the maximum of these sets. The other implication is obvious. In summary, $x \in \Omega_k$ if $E_k f(x)$ is the last expectation of f which is greater than λ. By writting each of the disjoint sets Ω_k as the union of cubes in $\mathcal{D}_k^n$, we have

$$\begin{aligned} |\{x \in \mathbf{R}^n : M_d f(x) > \lambda\}| &= \sum_{k \in \mathbf{Z}} |\Omega_k| \\ &< \sum_{k \in \mathbf{Z}} \int_{\Omega_k} \frac{E_k f}{\lambda}. \end{aligned}$$

Then by using the identity (2.4) :

$$\begin{aligned} |\{x \in \mathbf{R}^n : M_d f(x) > \lambda\}| &< \frac{1}{\lambda} \sum_{k \in \mathbf{Z}} \int_{\Omega_k} f \\ &\leqslant \frac{1}{\lambda} \|f\|_{\mathrm{L}^1(\mathbf{R}^n)}. \end{aligned}$$

The last inequality concludes the proof : M_d is weak $(1,1)$. □

It's quite obvious that M_d is strong (∞, ∞). Then by the Marcinkiewicz interpolation theorem we immedialty have the following corollary.

Corollary 2.15. *Let* $1 < p < \infty$. *The dyadic maximal function is strong* (p,p), *ie for all* $f \in \mathrm{L}^p(\mathbf{R}^n)$

$$\|M_d f\|_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim \|f\|_{\mathrm{L}^p(\mathbf{R}^n)}.$$

2.4. **Hardy-Littlewood maximal theorem**

Theorem 2.16 (Hardy-Littlewood maximal theorem)**.** *(i) Let* $1 < p \leqslant \infty$. *The Hardy-Littlewood maximal function* M_{HL} *is strong* (p,p), *ie for all* $f \in \mathrm{L}^p(\mathbf{R}^n)$

$$\|M_{\mathrm{HL}}\|_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim \|f\|_{\mathrm{L}^p(\mathbf{R}^n)}.$$

(ii) The Hardy-Littlewood maximal function M_{HL} *is weak* $(1,1)$.

The theorem is clear for $p=\infty$, and we already in the seen example 2.3 that the result isn't true for $p=1$. The Hardy-Littlewood maximal function is weak (∞,∞), so by the Marcinkiewicz interpolation theorem it suffices to prove the second part of the theorem.

Proof. Let f be an integrable function and $\lambda>0$. We may assume that f is a non-negative function (because $\mathrm{M}'_{\mathrm{HL}}|f| = M'_{\mathrm{HL}}f$). We are going to prove that

$$|\{x\in\mathbf{R}^n : M'_{\mathrm{HL}}f(x) > 4^n\lambda\}| \leqslant 2^n|\{x\in\mathbf{R}^n : M_d f(x) > \lambda\}|. \tag{2.5}$$

We recall that M'_{HL} is the maximal function defined on cubes. We saw that these two maximal functions are kind of equivalent (see inequalities (2.2)). By the theorem 2.14, we know that

$$|\{x\in\mathbf{R}^n : M_d f(x) > \lambda\}| \lesssim \frac{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}}{\lambda}. \tag{2.6}$$

Let's prove the inequality (2.5). As before, we are able to find some cubes $\{Q_i\}_i$ such that

$$\{x\in\mathbf{R}^n : M_d f(x) > \lambda\} = \bigcup_i Q_i$$

(see the proof of the theorem 2.14). Let Q_i^* be the cube with the same center as Q_i and whose sides are twice as long : $l(Q_i^*) = 2l(Q_i)$. We fix an $x\notin \cup_i Q_i^*$, and we denote by Q a cube centered at x. Let $k\in\mathbf{Z}$ be the integer such that $2^{k-1} < l(Q) < 2^k$. Then Q intersects at the maximum 2^n cubes in $\mathcal{D}_k^n$ (see figure 2). Let $R_1,\dots,R_m$ be these cubes, with $m\leqslant 2^n$.

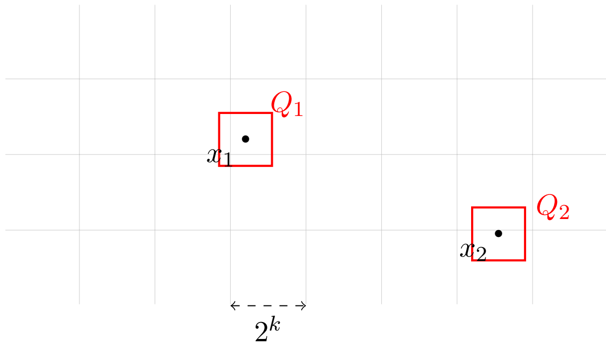

Figure 2

If one of these cubes is contained in a cube Q_i, we would have $x\in\cup_i Q_i^*$, which is false. In fact if there exist some indexes i,j such that $R_i\subset Q_j$, we would have on the on hand $x\in$ Q and on the other hand $R_i^*\subset Q_j^*$. But $l(Q)<2^k$, so $x\in R_i^*$. We finally got $x\in Q_j^*$. Hence the average of f on each cube $\{R_i\}_i$ is at most λ. Then we have

$$⨍_Q f = \frac{1}{|Q|}\sum_{i=1}^m \int_{Q\cap R_i} f \leqslant \frac{1}{|Q|}\sum_{i=1}^m 2^{kn} ⨍_{R_i} f \leqslant \frac{2^{kn}}{2^{n(k-1)}} m\lambda \leqslant 4^n\lambda.$$

So we just proved that

$$\{x\in\mathbf{R}^n : M'_{\mathrm{HL}} > 4^n\lambda\}\subset \bigcup_i Q_i^*.$$

Hence

$$|\{x\in\mathbf{R}^n : M'_{\mathrm{HL}}f(x) > 4^n\lambda\}| \leqslant 2^n\left|\bigcup_i Q_i\right| = 2^n|\{x\in\mathbf{R}^n : M_d f(x) > \lambda\}|$$

which is the inequality (2.5). By using (2.5) and (2.6) we finally have

$$|\{x\in\mathbf{R}^n : M'_{\mathrm{HL}}f(x) > \lambda\}| \lesssim \frac{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}}{\lambda},$$

which concludes the proof. ∎

3. Maximal Function

3.1. Almost everywhere convergence

There is a relationship between weak (p, q) inequalities and almost everywhere convergence, and it is given by the following result.

Lemma 3.1. *Let (X, μ) be a measure space, $1 \leqslant p, q < \infty$ and $\{T_t\}_{t\in\mathcal{A}}$ be a family of linear operators on $\mathrm{L}^p(X, \mu)$, with $\mathcal{A} \subset (0, \infty)$. Let $t_0 \in [0, \infty]$ be a limit point of $\mathcal{A}$. We introduce the* ***maximal operator*** *associated with the family $\{T_t\}_t$:*

$$T^*f : x \longmapsto T^*f(x) := \sup_{t\in\mathcal{A}} |T_t f(x)|.$$

If T^* is weak (p, q), then the set

$$\left\{ f \in L^p(X, \mu) : \lim_{t\to t_0} T_t f(x) = f(x) \text{ almost everywhere} \right\}$$

is closed in $\mathrm{L}^p(X, \mu)$.

Example 3.2. The maximal operator associated to the family $\{A_r|\cdot|\}_{r>0}$ is the Hardy-Littlewood maximal function.

With the same assumptions as in the previous lemma, we obtain the following theorem, which is a direct consequence from the sequential characterisation of closed spaces.

Theorem 3.3. *Futhermore, if we assume that there exist a dense subspace $D \subset \mathrm{L}^p(X, \mu)$ such that for all $f \in D$*

$$\lim_{t\to t_0} T_t f(x) = f(x) \text{ for } \mu\text{-a.e } x \in \mathrm{X}.$$

Then for all $f \in \mathrm{L}^p(X, \mu)$

$$\lim_{t\to t_0} T_t f(x) = f(x) \text{ for } \mu\text{-a.e } x \in \mathrm{X}.$$

Let's demonstrate the lemma.

Proof. Let $\{f_n\}_{n\in\mathbf{N}}$ be a sequence of fucntions which converges to another function f in $\mathrm{L}^p(X, \mu)$ norm, and such that for all $n \in \mathbf{N}$ and for μ-almost every $x \in X$, $\lim_{t\to t_0} T_t f_n(x) = f_n(x)$. We are going to show that $\lim_{t\to t_0} T_t f(x) = f(x)$ for μ-almost every $x \in X$. We temporarily fix a real $r > 0$, and we will see that the quantity $\mu(\{x \in \mathrm{X} : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \lambda\})$ is equal to 0. In fact we have :

$$\begin{aligned}\mu(\{x \in \mathrm{X} : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \lambda\}) &= \mu(\{x \in \mathrm{X} : \limsup_{t\to t_0} |T_t(f - f_n)(x) - (f - f_n)(x)| > \lambda\}) \\ &\leqslant \mu\left(\{x \in \mathrm{X} : T^*(f - f_n)(x) > \lambda/2\}\right) \\ &+ \mu\left(\{x \in \mathrm{X} : |(f - f_n)(x)| > \lambda/2\}\right).\end{aligned}$$

We assumed that the T^*0 is weak (p, q), so the first term can be bounded by

$$\lesssim \left(\frac{\|f\|_{\mathrm{L}^p(X,\mu)}}{\lambda}\right)^q$$

which tends to 0 as $n \to \infty$. The second term can be bounded by using the inequality of Markov :

$$\leqslant \left(\frac{2\|f\|_{\mathrm{L}^p(X)}}{\lambda}\right)^p$$

and this bound also tends to 0 as $n \to \infty$. Finally $\mu(\{x \in \mathrm{X} : \limsup_{t\to t_0} |T_t f(x) - f(x)| > \lambda\}) = 0$ for all $\lambda > 0$. To conclude, it suffices to write

$$\mu(\{x \in \mathrm{X} : \limsup_{t\to t_0} |T_t f(x) - f(x)| > 0\}) \leqslant \sum_{k=1}^{\infty} \mu(\{x \in \mathrm{X} : \limsup_{t\to t_0} |T_t f(x) - f(x)| > 1/k\}) = 0.$$

$\square$

3.2. **Lebesgue differentiation theorem**

The operators $\{E_k\}_{k\in\mathbf{Z}}$ are linear, and the maximal operator associated is

$$f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n) \longmapsto \sup_{k\in\mathbf{Z}} |E_k f| \,.$$

Since for all $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$, $\sup_{k\in\mathbf{Z}} |E_k f| \leqslant \mathrm{M}_d f$, this maximal operator is, as the dyadic maximal function (theorem 2.14), weak $(1,1)$. Futhermore, we know that if f is continuous then

$$\lim_{k\to-\infty} E_k f(x) = f(x) \text{ a.e.}$$

The subspace of continuous function on $\mathbf{R}^n$ is dense in $\mathrm{L}^1(\mathbf{R}^n)$, so by the theorem 3.3 we have the following result :

Proposition 3.4. *For all $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$,*

$$\lim_{k\to-\infty} E_k f(x) = f(x) \text{ a.e.}$$

Remarque 1. The proposition holds for $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$ because if $f \in \mathrm{L}^1(\mathbf{R}^n)$, then $f\chi_Q \in \mathrm{L}^1(\mathbf{R}^n)$ for any dyadic cube Q, so for almost every $x \in Q$ and finally for almost every $x \in \mathbf{R}^n$.

We also have a continuous analog of the proposition 3.4, knew as the Lebesgue differentiation theorem. We know that M_{HL} is weak $(1,1)$. Again, by considering the maximal operator

$$f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n) \longmapsto \sup_{r>0} |A_r f|,$$

we can apply the theorem 3.3 and obtain the followgin result, knew as to be true for the continuous functions.

Theorem 3.5 (Lebesgue Differentiation Theorem)**.** *For all $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$,*

$$\lim_{r\to 0^+} \fint_{B_r} f(x-y)\mathrm{d}y = f(x) \text{ a.e.}$$

3.3. **The Stein's maximal principle**

We saw that some weak bounds on an maximal operator can bring some convergence results. In fact the reverse can be true under some assumptions : it's the Stein maximal principle.

3.3.1. **Statement and applications**

Theorem 3.6. *Let $\{\mu_j\}_{j\in\mathbf{N}}$ be a sequence of finite Borel measures on $\mathbf{R}^n$. We assume that they are all supported on a fixed compact $Q_0 := [-1/2, 1/2)^n$, ie*

$$\forall\, j \in \mathbf{N}, \quad \mathrm{supp}(\mu_j) \subset Q_0.$$

Let M be a maximal function of the form

$$Mf(x) = \sup_{j\in\mathbf{N}} |f| * \mu_j(x) \tag{3.1}$$

where $f \in \mathrm{L}^p(\mathbf{R}^n)$, $1 \leqslant p < \infty$ and $x \in \mathbf{R}^n$. Assume for each $f \in \mathrm{L}^p(\mathbf{R}^n)$ we have

$$Mf(x) < \infty, \quad \text{for all } x \text{ lying in a set of positive measure.}$$

Then M is weak (p,p).

The assumptions about the form of the maximal function and about the measures are necessary. In fact the result isn't true all the time. It suffices to consider the the family of linear operator

$$T_k : \left| \begin{array}{ccc} \mathrm{L}^1(\mathbf{R}) & \longrightarrow & \mathrm{L}^1(\mathbf{R}) \\ f & \longmapsto & \chi_{[k,k+1]} \int_0^1 f \end{array} \right.$$

for $k \in \mathbf{N}$. Here for $f \in \mathrm{L}^1(\mathbf{R})$ and $x \in \mathbf{R}$ we clearly have

$$T^* f(x) := \sup_{k \in \mathbf{N}} |T_k f(x)| = \left| \int_0^1 f \right| \leqslant \|f\|_{\mathrm{L}^1(\mathbf{R})} < \infty.$$

However T^* is not weak $(1,1)$. If it was the case, we would have for all $\lambda > 0$

$$|\{x \in \mathbf{R} : T^* f(x) > \lambda\}| = \left|\left\{x \in \mathbf{R} : \left| \int_0^1 f \right| > \lambda\right\}\right| \leqslant \frac{\|f\|_{\mathrm{L}^1(\mathbf{R})}}{\lambda}.$$

By choosing $\lambda := \left| \int_0^1 f \right| /2$ (and f such that $\lambda \neq 0$), we obtain a contradiction, the left term is equal to ∞.

We can write an anolog of the Hardy-Littlewood maximal function as a function of the form (3.1). We introduce the operator

$$M^* f(x) := \sup_{\substack{0<r<1/2 \\ r \in \mathbf{Q}}} \fint_{B_r} |f(x-y)| \mathrm{d}y,$$

where $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$. Let $\{r_i\}_{i \in \mathbf{N}}$ be a sequence of rationals such that $\lim_{i \to \infty} r_i = 0$. By the theorem 3.5 we know that

$$\lim_{i \to \infty} \fint_{B_{r_i}} f(x-y) \mathrm{d}y = f(x)$$

for almost every $x \in \mathbf{R}^n$. In particular, this implies that $M^* f(x) < \infty$ almost everywhere, whenever $f \in \mathrm{L}^1(\mathbf{R}^n)$, so by the Stein maximal principle asserts that M^* is weak $(1,1)$. We can show that M^* is weak $(1,1)$ (and strong (p,p)) if and only if M_{HL} is. In summary,

$$\boxed{\text{Lebesgue Differentiation Theorem} \iff \text{Stein's Maximal Principle.}}$$

Another application is the study of convergence problem for Fourier integrals. We already talk about it in the introduction, see theorem 1.4 and theorem 1.5. We can use the Stein's maximal principle and the theory of the maximal function to study some convergence problems.

Definition 3.7. Let $f \in \mathrm{L}^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant 2$. We define the **Carleson maximal operator** by

$$\mathcal{C} f(x) := \sup_{R>0} |S_R f(x)|$$

for $x \in \mathbf{R}^n$.

In view of the theorem 3.3, the almost everywhere convergence questions for Fourier integrals are equivalent to a weak (p,p) bound for the Carleson maximal operator $\mathcal{C}$.

Proposition 3.8. *Let $1 \leqslant p \leqslant 2$. The following are equivalent :*

(i) For all $f \in \mathrm{L}^p(\mathbf{R}^n)$

$$\lim_{R \to \infty} S_R f(x) = f(x), \quad \textit{for almost every } x \in \mathbf{R}^n.$$

(ii) $\mathcal{C}$ is weak (p,p).

Proving some bounds for the operator $\mathcal{C}$ is a very difficult task. We often consider an analog version of the Carleson maximal operator, which is easier to bound, and still gives us some interesting results.

Definition 3.9. Let $f \in \mathrm{L}^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant 2$. We define the **lacunary Carleson maximal operator** by

$$\mathcal{C}_{\mathrm{lac}} f(x) := \sup_{k \in \mathbf{N}} |S_{2^k} f(x)|$$

for $x \in \mathbf{R}^n$.

Let's move on the proof of the Stein's maximal principle.

3.4. **Proof**

Before proving the Stein's maximal principle, we present three technical lemmas, the proofs of these results can be found in [6].

Lemma 3.10 (Local reduction). *Let M be a maximal function of the form* (3.1)*, and $1 \leqslant p < \infty$. Assume that for all $\lambda > 0$ and for all $f \in \mathrm{L}^p(\mathbf{R}^n)$ with* $\operatorname{supp}(f) \subset Q_0^* := [-1,1)^n$ *we have*

$$|\{x \in Q_0 : Mf(x) > \lambda\}| \lesssim \left(\frac{\|f\|_{\mathrm{L}^p(\mathbf{R}^n)}}{\lambda}\right)^p.$$

Then M is weak (p,p).

Lemma 3.11 (Random translation). *Let $E \subset Q_0$ be a measurable set with $|E| > 0$. Then there exist some vectors $x_1, \dots, x_J \in Q_0^*$ with $J \lesssim 1/|E|$, such that*

$$\left| Q_0 \cap \bigcup_{j=1}^{J} (E + x_j) \right| \geqslant 1/2.$$

Lemma 3.12 (Borel–Cantelli-type lemma). *Let $(F_k)_{k \in \mathbf{N}^*}$ be a sequence of measurables subsets of $\mathbf{R}^n$ such that*

$$\sum_{k=1}^{\infty} |F_k| = +\infty.$$

Then there exists a sequence of vectors $(x_k)_{k \in \mathbf{N}^*}$ such that

$$\limsup_{k \to +\infty} (F_k + x_k) = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} (F_j + x_j) = \mathbf{R}^n \backslash N,$$

for some null set $N \subset \mathbf{R}^n$. Almost every $x \in \mathbf{R}^n$ lies in infinitely many of the translated sets $E_k + x_k$.

We can pass to the proof of the Stein's maximal principle.

Proof. We are. going to prove the theorem via the negation. Assume that M is not strong (p,p). By the lemma 3.10, there exists a sequence $\{g_k\} \in \mathrm{L}^p(\mathbf{R}^n)$ and real values $\lambda_k > 0$ such that

$$g_k > 0 \quad \text{and} \quad |\{x \in Q_0 : Mg_k(x) > \lambda_k\}| \geqslant \left(\frac{k2^k}{\lambda_k} \|g_k\|_{\mathrm{L}^p(\mathbf{R}^n)}\right)^p$$

for all $k \in \mathbf{N}$. By homogeneity, we can replace g_k by $\frac{\lambda_k}{k} g_k$, and so without loss of generality we may assume that

$$|\{x \in Q_0 : Mg_k(x) > k\}| \geqslant 2^{kp} \|g_k\|_{\mathrm{L}^p(\mathbf{R}^n)}^p \tag{3.2}$$

for all $k \in \mathbf{N}$. Let fix $k \in \mathbf{N}$, and denote

$$E_k := \{x \in Q_0 : Mg_k(x) > k\}.$$

In view of the above, $|E_k|$ is large relative to $\|g_k\|^p_{\mathrm{L}^p(\mathbf{R}^n)}$, it may still have small measure in absolute terms. We are going to apply the lemma 3.11 on the sets E_k to ensure that $|E_k|$ is large enough. Let $x^k_1, \dots x^k_{J_k}$ be the sequence given by the lemma with $J_k \lesssim 1/|E_k|$. Let

$$F_k := Q_0 \cap \bigcup_{j=1}^{J_k} E_k + x_{k,j}$$

and

$$f_k(x) := \sup_{1 \leqslant j \leqslant J_k} \tilde{g}_{k,j}(x), \quad \forall x \in \mathbf{R}^n$$

where $\tilde{g}_{k,j}(x) := g_k(x - x_{k,j})$ (the translate). Note that for $x \in E_k + x_k$ we have by (3.2)

$$Mf_k(x) \geqslant \sup_{1 \leqslant j \leqslant J_k} M\tilde{g}_{k,j}(x) \geqslant k.$$

So

$$F_k \subset \{x \in Q_0 : Mf_k(x) \geqslant k\}.$$

By applying the lemma 3.11 we see that

$$|\{x \in Q_0 : Mf_k(x) > k\}| \geqslant |F_k| \geqslant 1/2.$$

On the other hand, the L^p-norms remain small. In fact, since for all $x \in \mathbf{R}^n$ $|f_k(x)|^p \leqslant \sum_{j=1}^{J_k} |\tilde{g}_{k,j}(x)|^p$, it follows that

$$\|f_k\|^p_{\mathrm{L}^p(\mathbf{R}^n)} \leqslant \sum_{j=1}^{J_k} \|\tilde{g}_{k,j}\|^p_{\mathrm{L}^p(\mathbf{R}^n)} = J_k \|g_k\|^p_{\mathrm{L}^p(\mathbf{R}^n)},$$

and then

$$\|f_k\|^p_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim 2^{-kp}. \tag{3.3}$$

Now we are going to use the functions f_k to build another function f which will be very bad for the maximal function. The sets satisfy $|F_k| \geqslant 1/2$ and clearly $\sum_{k=1}^{\infty} |F_k| = \infty$. So the lemma 3.12 gives us a sequence of translates $(x_k)_{k \in \mathbf{N}^*}$ such that

$$\limsup_{k \to +\infty} (F_k + x_k) = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} (F_j + x_j) = \mathbf{R}^n \backslash N,$$

where $N \subset \mathbf{R}^n$ is a null set. We define f by

$$f(x) := \sup_{k \in \mathbf{N}} f_k(x), \quad \forall x \in \mathbf{R}^n.$$

Then

$$\begin{aligned}\{x \in Q_0 : Mf(x) = \infty\} &= \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \{x \in Q_0 : Mf(x) \geqslant j\} \\ &\supset Q_0 \cap \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} (F_j + x_j) = Q_0 \backslash N.\end{aligned}$$

Since $|N| = 0$, $Mf(x) = \infty$ for almost every $x \in Q_0$. On the other hand, with (3.3) we have

$$\|f\|^p_{\mathrm{L}^p(\mathbf{R}^n)} \leqslant \sum_{k=1}^{\infty} \|f_k\|^p_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim \sum_{k=1}^{\infty} < \infty,$$

so we get a contradiction. □

3.5. Other geometric maximal function

We give here some examples of geometric maximal function and their main results. We defined a maximal function on balls (and on cubes), with the Hardy-Littlewood maximal function. We can wonder what happend if we consider some maximal functions on other geometric shape. In fact it becomes very complicated very quickly.

3.5.1. The Strong maximal function

Let's begin by define a maximal function on rectangles with sides parallel to the coordinate axes. This function is called the **strong maximal function**, and its defined as below.

Definition 3.13. For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$, we define the **strong maximal function** by

$$M_{\mathrm{st}}f(x) = \sup_{r_1,\dots,r_n>0} \fint_{\prod_{i=1}^n[-r_i,r_i]} |f(x-y)|\mathrm{d}y$$

where $x \in \mathbf{R}^n$.

M_{st} has not the same behaviour as M_{HL}. In fact, we can show that for $1 < p \leqslant \infty$, M_{st} is strong (p,p), as M_{HL}. However the strong maximal function **is not** weak $(1,1)$. We can find a proof of these results in [6].

3.5.2. The spherical maximal function

Let σ be the surface area measure on $\mathbf{S}^{n-1}$. We define a maximal function on the sphere $\mathbf{S}^{n-1}$.

Definition 3.14. For $f \in \mathcal{C}^0(\mathbf{R}^n)$, we define the **spherical maximal function** by

$$M_\sigma f(x) := \sup_{r>0} \fint_{\mathbf{S}^{n-1}} |f(x-ry)|\mathrm{d}\sigma(y)$$

where $x \in \mathbf{R}^n$.

We have the following result for $n \geqslant 3$.

Theorem 3.15 (Stein, 1976)**.** *Let $f \in \mathrm{L}^p(\mathbf{R}^n)$, then*

$$\|M_\sigma f\|_{\mathrm{L}^p(\mathbf{R}^d)} \lesssim_p \|f\|_{\mathrm{L}^p(\mathbf{R}^d)}$$

for all $\frac{n}{n-1} < p \leqslant \infty$.

This result is not very difficult to prove, it is based on geometrical estimations and on some basic tools of harmonic analysis (discretisation, duality, ...). The result for $n = 2$ have been proved ten years later by Bourgain.

Theorem 3.16 (Bourgain, 1986)**.** *Let $f \in \mathrm{L}^p(\mathbf{R}^2)$, then*

$$\|M_\sigma f\|_{\mathrm{L}^p(\mathbf{R}^2)} \lesssim_p \|f\|_{\mathrm{L}^p(\mathbf{R}^d)}$$

for all $2 < p \leqslant \infty$.

Let us explain what goes wrong with the case $n = 2$. In the proof of the Stein's theorem, we have to integrate the function $t \longmapsto (1-t^2)^{\frac{n-3}{2}}$ on $[-1,1]$, but it fails if $n = 2$. In our problem it comes from the value of the intersection of tangent circles of a certain thickness δ. In fact if C_1, C_2 are such circles, we would have

$$|C_1 \cap C_2| \sim \delta^{3/2}.$$

See the reference [7] for more details.

3.5.3. The Nikodym maximal function

Let's now consider the set $\mathcal{R}_N^k$ for each $N, k \in \mathbf{N}^*$ of all rectangles in $\mathbf{R}^n$ centered at the origin and with the dimension $\underbrace{a \times \cdots \times a}_{n-k} \times \underbrace{aN \times \cdots \times aN}_{k}$, for any $a > 0$, and with arbitrary orientation.

Definition 3.17. For $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$, we define the k**-plane Nikodym maximal function** by

$$M_{\mathcal{R}_N^k} f(x) := \sup_{R \in \mathcal{R}_N^k} \fint_R |f(x-y)| \mathrm{d}y$$

where $x \in \mathbf{R}^2$.

Y. Choi, Y. Koh and J. Lee did the followgin conjecture in [2], for $f \in \mathrm{L}^p(\mathbf{R}^n)$.

Conjecture 3.18. For an N large enough :

$$\begin{cases} \|M_{\mathcal{R}_N^k} f\|_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim_{p,\varepsilon} N^{\frac{n}{p}-k+\varepsilon} \|f\|_{\mathrm{L}^p(\mathbf{R}^n)} & \text{if } 1 < p \leqslant \frac{n}{k} \\ \|M_{\mathcal{R}_N^k} f\|_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim_{p,\varepsilon} N^{\varepsilon} \|f\|_{\mathrm{L}^p(\mathbf{R}^n)} & \text{if } \frac{n}{k} \leqslant p \leqslant \infty \end{cases}$$

for all $\varepsilon > 0$.

When $k = 1$, it was shown by T. Tao in [10] that this conjecture is equivalent to the Kekeya set conjecture, which is the following (we will not detail the theory about the Minkowsi dimension here, for more information about it, see [3]).

Conjecture 3.19 (Kakeya set)**.** Define a Kakeya set to be any subset $E \subset \mathbf{R}^n$ which contains a unit line segment in each direction. Then all Kakeya sets have Minkowski dimension n.

When $n \geqslant 3$, there are only partial results and when $k \geqslant 2$ there is no known result. As we can see, by considering only rectangles with no assumption on their orientation, the problem stills open.

4. The Hilbert Transform

We are going to study here a fundamental operator in analysis, the Hilbert transform. First we introduce the notion of Fourier multipliers.

4.1. The multipliers

Definition 4.1. Given $m \in \mathrm{L}^\infty(\widehat{\mathbf{R}}^n)$, we define the associated Fourier multiplier operator T_m by

$$T_m : \left| \begin{array}{ccc} \mathrm{L}^2(\mathbf{R}^n) & \longrightarrow & \mathrm{L}^2(\mathbf{R}^n) \\ f & \longmapsto & \mathcal{F}^{-1}(m \cdot \mathcal{F}f). \end{array} \right.$$

By the Plancherel's theorem, T_m is well defined on $\mathrm{L}^2(\mathbf{R}^n)$ and he's bounded :

$$\|T_m f\|_{\mathrm{L}^2(\mathbf{R}^n)} = \|m \cdot \hat{f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)} \leqslant \|m\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} \|f\|_{\mathrm{L}^2(\mathbf{R}^n)}. \tag{4.1}$$

The Fourier multiplier of a function $m \in \mathrm{L}^\infty(\widehat{\mathbf{R}}^n)$ is the only operator such that

$$(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$$

for all $\xi \in \widehat{\mathbf{R}}^n$. Hence the operator T_m acts as a filter on the frequency of the function f. For example, if $m = \chi_{[1,\infty)}$, the associated Fourier multiplier will be a high-pass filter. If $f \in \mathrm{L}^2(\mathbf{R}^n)$, then

$$T_{\chi_{[1,\infty)}} f : x \in \mathbf{R}^n \longmapsto \int_1^\infty \hat{f}(\xi) e^{2i\pi\xi\cdot x} \mathrm{d}\xi.$$

Since T_m is a bounded operator on $\mathrm{L}^2(\mathbf{R}^n)$, a legitimate question is to ask about its operator norm $\|T_m\|_{\mathrm{L}^2(\mathbf{R}^n)\to\mathrm{L}^2(\mathbf{R}^n)}$. In fact we know the exact value of this norm.

Lemma 4.2. *If* $m \in \mathrm{L}^\infty(\widehat{\mathbf{R}}^n)$ *then*

$$\|T_m\|_{\mathrm{L}^2(\mathbf{R}^n)\to\mathrm{L}^2(\mathbf{R}^n)} = \|m\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)}.$$

Proof. By the inequality (4.1), we know that

$$\|T_m\|_{\mathrm{L}^2(\mathbf{R}^n)\to\mathrm{L}^2(\mathbf{R}^n)} \leqslant \|m\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)}.$$

We fix an $\varepsilon > 0$ and let $\mathcal{A}$ be measurable subset of $\{\xi \in \widehat{\mathbf{R}}^n : |m(\xi)| > \|m\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} - \varepsilon\}$ whose measure is finite and positive (in fact $|\mathcal{A}|$ isn't alaways finised, for example if m is a constant function). Let f be the function in $\mathrm{L}^2(\mathbf{R}^n)$ such that $\hat{f} = \chi_{\mathcal{A}}$. Then

$$\|T_m f\|_{\mathrm{L}^2(\mathbf{R}^n)} = \|m \cdot \hat{f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)} > (\|m\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} - \varepsilon)\|f\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)}.$$

So for all $\varepsilon > 0$

$$\|T_m\|_{\mathrm{L}^2(\mathbf{R}^n)\to\mathrm{L}^2(\mathbf{R}^n)} \geqslant \|m\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} - \varepsilon,$$

which concludes the proof. □

4.2. Definition

We would like to define the Hilbert transform of a function $f \in \mathcal{S}(\mathbf{R})$ as the convolution of f and the function $t \to 1/\pi t$, ie

$$\mathrm{H}f : x \in \mathbf{R} \longmapsto \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(x-y)}{y} \mathrm{d}y.$$

The issue is that this object is not well defined, even on $\mathcal{S}(\mathbf{R})$, because of the singularity on 0. A solution to switch the function $t \longmapsto 1/t$ by the tempered distribution $\mathrm{vp}(1/x)$, defined as

$$\mathrm{vp}(1/x) : \left| \begin{array}{ccc} \mathcal{D}(\mathbf{R}) & \longrightarrow & \mathbf{R} \\ \varphi & \longmapsto & \langle \mathrm{vp}(1/x), \varphi \rangle := \lim\limits_{\varepsilon\to 0} \displaystyle\int_{|x|\geqslant\varepsilon} \frac{\varphi(y)}{y} \mathrm{d}y. \end{array} \right.$$

Definition 4.3. Given a function $f \in \mathcal{S}(\mathbf{R})$, we define its **Hilbert transform** $\mathrm{H}f$ as

$$\mathrm{H}f(x) := \frac{1}{\pi}\mathrm{vp}(1/x) * f(x),$$

where $x \in \mathbf{R}$.

Let $f \in \mathrm{L}^p(\mathbf{R})$, $1 \leqslant 2 \leqslant p$ and $x \in \mathbf{R}$. We would like to study the behavior when $R \to \infty$ of the integral

$$S_R f := T_{\chi_{[-R,R]}} f = \int_{-R}^{R} \hat{f}(\xi) e^{2i\pi\xi\cdot x} \mathrm{d}\xi,$$

and determine when it converges to $f(x)$, ie recovering a function f from its Fourier transform $\hat{f}$. We will see that there is a clear link between the operator S_R and the Hilbert transform. In fact for all $a < b$,

$$S_R = \frac{i}{2}\left(m_R \mathrm{H} m_{-R} - m_R \mathrm{H} m_{-R}\right) \tag{4.2}$$

where m_R is a modulation operator defined by

$$m_R f(x) := e^{2i\pi Rx} f(x).$$

This last result gives to the Hilbert transform even more interest. Let see another equivalent way to define the Hilbert transform.

Definition 4.4. For all $t > 0$ we define the **conjugate Poisson kernel** $Q_t : \mathbf{R} \to \mathbf{R}$ by

$$Q_t : x \in \mathbf{R} \longmapsto Q_t(x) := \frac{1}{\pi}\frac{x}{t^2 + x^2}.$$

Proposition 4.5. *Given a function $f \in \mathcal{S}(\mathbf{R})$ its Hilbert transform is also defined as*

$$\mathrm{H}f = \lim_{t \to 0^+} Q_t * f. \tag{4.3}$$

Proof. It suffices to show that

$$\lim_{t \to 0^+} Q_t = \frac{1}{\pi}\mathrm{vp}(1/x)$$

in $\mathcal{S}'(\mathbf{R})$. The functions $\{Q_t\}$ are in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R})$, so they define distributions. Let φ be a test function in $\mathcal{S}(\mathbf{R})$. We need to show that

$$\lim_{t \to 0^+} \langle Q_t, \varphi \rangle = \frac{1}{\pi}\langle \mathrm{vp}(1/x), \varphi \rangle.$$

We introduce some truncated versions of the inverse functions, for all $\varepsilon > 0$ we define

$$\psi_\varepsilon : x \in \mathbf{R} \longmapsto \frac{1}{x}\chi_{\{|x| > \varepsilon\}}.$$

These functions define some tempered distributions. It's clear that

$$\lim_{\varepsilon \to 0^+} \psi_\varepsilon = \mathrm{vp}(1/x)$$

in $\mathcal{S}'(\mathbf{R})$. Then it suffices to prove that

$$\lim_{t \to 0^+} \langle \pi Q_t - \psi_t, \varphi \rangle = 0.$$

We have

$$\begin{aligned}\langle \pi Q_t - \psi_t, \varphi\rangle &= \int_{\mathbf{R}} \frac{x\varphi(x)}{t^2+x^2}\mathrm{d}x - \int_{|x|>t} \frac{\varphi(x)}{x}\mathrm{d}x \\ &= \int_{|x|\leqslant t} \frac{x\varphi(x)}{t^2+x^2}\mathrm{d}x + \int_{|x|>t} \varphi(x)\left(\frac{x}{t^2+x^2} - \frac{1}{x}\right)\mathrm{d}x \\ &= \int_{|y|\leqslant 1} \frac{y\varphi(yt)}{1+y^2}\mathrm{d}y - \int_{|y|>1} \frac{\varphi(yt)}{y(1+y^2)}\mathrm{d}y.\end{aligned}$$

Then we apply the dominated convergence theorem

$$\lim_{t\to 0^+} \langle \pi Q_t - \psi_t, \varphi\rangle = \varphi(0)\left(\int_{|y|\leqslant 1} \frac{y}{1+y^2}\mathrm{d}y - \int_{|y|>1} \frac{1}{y(1+y^2)}\mathrm{d}y\right) = 0.$$

□

4.3. Main results

Lemma 4.6. *For all $\xi \in \widehat{\mathbf{R}}$, in the sense of distributions we have*

$$\left(\frac{1}{\pi}\mathrm{vp}(1/x)\right)^{\wedge}(\xi) = -i\mathrm{sgn}(\xi).$$

Proof. We saw that $\frac{1}{\pi}\mathrm{vp}(1/x) = \lim_{t\to 0^+} Q_t$. So by the continuity of the Fourier transform on $\mathcal{S}'(\mathbf{R})$, we have for all $\xi \in \widehat{\mathbf{R}}$

$$\left(\frac{1}{\pi}\mathrm{vp}(1/x)\right)^{\wedge}(\xi) = (\lim_{t\to 0^+} Q_t)^{\wedge}(\xi) = \lim_{t\to 0^+} \widehat{Q_t}(\xi).$$

By using the inverse Fourier transform we can easily show that $\widehat{Q_t}(\xi) = -i\mathrm{sgn}(\xi)e^{-2\pi t|\xi|}$, the lemma follows by taking the limit as $t \to 0^+$. □

This lemma gives us an expression of the Fourier transform of the Hilbert transform of a Schwartz function. With the following expression, we can easily obtain the identiy (4.2).

Proposition 4.7. *Let f be a function in $\mathcal{S}(\mathbf{R})$. The Fourier tranform of $\mathrm{H}f$ is given for all $\xi \in \widehat{\mathbf{R}}$ by*

$$(\widehat{\mathrm{H}f})(\xi) = -i\mathrm{sgn}(\xi)\hat{f}(\xi).$$

This expression lets us define the Hilbert transform on $\mathrm{L}^2(\mathbf{R})$. Futhermore we have the following corollary.

Corollary 4.8. *Let f be a function in $\mathrm{L}^2(\mathbf{R})$. Then we have the following results.*

(*i*) $\mathrm{H}f \in \mathrm{L}^2(\mathbf{R})$ *and*

$$\|\mathrm{H}f\|_{\mathrm{L}^2(\mathbf{R})} = \|f\|_{\mathrm{L}^2(\mathbf{R})}.$$

(*ii*)

$$\mathrm{H}(\mathrm{H}f) = -f.$$

(*iii*) *If $g \in \mathrm{L}^2(\mathbf{R})$ then*

$$\int_{\mathbf{R}} \mathrm{H}f \cdot g = -\int_{\mathbf{R}} f \cdot \mathrm{H}g.$$

Proof. (*i*) We apply the Plancherel's theorem and the proposition 4.7,

$$\|\mathrm{H}f\|_{\mathrm{L}^2(\mathbf{R})} = \|\widehat{\mathrm{H}f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}})} = \|\hat{f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}})} = \|f\|_{\mathrm{L}^2(\mathbf{R})}.$$

(*ii*) We fixed a real $\xi \in \widehat{\mathbf{R}}$.

$$(\mathrm{H}(\mathrm{H}f))^\wedge(\xi) = -i\mathrm{sgn}(\xi)(\widehat{\mathrm{H}f})(\xi) = (-i\mathrm{sgn}(\xi))^2\hat{f}(\xi) = -\hat{f}(\xi).$$

Again, we conclude with the Plancherel's theorem.

(*iii*)

$$\int_{\mathbf{R}} \mathrm{H}f \cdot g = \int_{\widehat{\mathbf{R}}} (\widehat{\mathrm{H}f})\hat{g} = \int_{\widehat{\mathbf{R}}} -i\mathrm{sgn}(\xi)\hat{f}(\xi)\hat{g}(\xi)\mathrm{d}\xi = \int_{\mathbf{R}} \hat{\hat{f}}(x)(\widehat{\widehat{\mathrm{H}g}})(x)\mathrm{d}x = -\int_{\mathbf{R}} f \cdot \mathrm{H}g.$$

□

Theorem 4.9. *Let* $1 \leqslant p < \infty$. *The Hilbert transform* H *can be extended on* $\mathrm{L}^p(\mathbf{R})$. *Furthermore :*

(*i*) *(Riesz)* H *is strong* (p,p) *:*

$$\|\mathrm{H}f\|_{\mathrm{L}^p(\mathbf{R})} \lesssim \|f\|_{\mathrm{L}^p(\mathbf{R})}$$

for all $f \in \mathrm{L}^p(\mathbf{R})$.

(*ii*) *(Kolmogorov)* H *is weak* $(1,1)$

$$|\{x \in \mathbf{R} : |\mathrm{H}f(x)| > \lambda\}| \lesssim \frac{\|f\|_{\mathrm{L}^1(\mathbf{R})}}{\lambda}$$

for all $f \in \mathrm{L}^1(\mathbf{R})$, $\lambda > 0$.

We will prove the theorem 4.9 in the next section, as a special case to another theorem.

5. Singular Integrals and the Calderón-Zygmund Theorem

We are going to study a more general theory which implies the results about the Hilbert transform : the study of singular integrals and in particular the Calderón Zygmund theorem.

5.1. Introduction and application to Hilbert transform

Given a tempered distribution $K \in \mathcal{S}'(\mathbf{R}^n)$, we will consider the associated convolution operator

$$K * f \tag{5.1}$$

for $f \in \mathcal{S}(\mathbf{R}^n)$.

Theorem 5.1 (Calderón-Zygmund)**.** *Let K be a tempered distribution. Assume that :*

- *(i) K coincides with a locally integrable function on $\mathbf{R}^n \backslash \{0\}$;*
- *(ii) there exists a constant A such that for all $\xi \in \widehat{\mathbf{R}}^n$*

$$|\hat{K}(\xi)| \leqslant A, \tag{5.2}$$

- *(iii) (Hörmander condition) there exists a constant B such that*

$$\sup_{y \in \mathbf{R}^n} \int_{|x|>2y} |K(x-y) - K(x)| \mathrm{d}x \leqslant B. \tag{5.3}$$

Then the convolution operator, defined initially on $\mathcal{S}(\mathbf{R}^n)$, is weak $(1,1)$ and strong (p,p) for $1 < p < \infty$.

An continuous linear map $\mathcal{S}(\mathbf{R}^n) \longrightarrow \mathcal{C}^0(\mathbf{R}^n)$ of the form $f \longmapsto K*f$ who respects these three condtions is called a **Calderón-Zygmund operator**. The condition (5.3) admits a stronger version, but often easier to apply.

Proposition 5.2. *If $K \in \mathcal{S}'(\mathbf{R}^n)$ coincides on $\mathbf{R}^n \backslash \{0\})$ with a function $\tilde{K} \in \mathcal{C}^1(\mathbf{R}^n \backslash \{0\}$ and assume*

$$|\nabla K(x)| \lesssim |x|^{-n-1}, \quad \forall x \in \mathbf{R}^n \backslash \{0\}. \tag{5.4}$$

Then the condition (5.3) holds.

Proof. The proof is an application of the mean value theorem, see the details on [5]. □

Before presenting the useful tools to prove the theorem (especially the Calderón–Zygmund decomposition), let us see how it can be applied to prove the theorem 4.9 on the Hilbert transform. Let's check the three conditions

- (i) The kernel considered here is the principal value $\frac{1}{\pi}\mathrm{vp}(1/x)$, which is well a tempered distribution. This principal value coincides with the function $t \longmapsto 1/\pi t$ on $\mathbf{R}^*$, and it's a function in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R})$.
- (ii) By the lemma 4.6 we know that for all $\xi \in \widehat{\mathbf{R}}$

$$\left| \left(\frac{1}{\pi} \mathrm{vp}(1/x) \right)^{\wedge} (\xi) \right| = |-i\mathrm{sgn}(\xi)| \leqslant 1,$$

  so the condition (5.2) holds.
- (iii) By the proposition 5.2, the assumption (5.3) is trivially verified.

Then we can conclude that the Hilbert transform is weak $(1,1)$ and strong (p,p) for $1 < p < \infty$, the theorem 4.9 is proved. We will see another application of this theorem in the next section. Let's move to the proof.

5.2. **The Calderón–Zygmund decomposition**

Given an integrable function $f \in \mathrm{L}^1(\mathbf{R}^n)$, the main idea of the proof is to break f into two parts, a good part g and a bad part b, where :

- $f = g + b$;
- g is essentially **bounded**;
- b is unbounded but has a **small support** and has a **zero mean**.

Let us demonstrate two lemmas which explain the interest of this decomposition. The first one is for the good part g and the second one is for the bad part b.

Lemma 5.3. *Suppose $K \in \mathcal{S}'(\mathbf{R}^n)$ satisfies that $\hat{K}$ coincides with $\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)$-function. Then for all $f \in \mathcal{S}(\mathbf{R}^n)$, $K * f \in \mathrm{L}^2(\mathbf{R}^n)$ and*

$$\|K * f\|_{\mathrm{L}^2(\mathbf{R}^n)} \leqslant \|\hat{K}\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} \|f\|_{\mathrm{L}^2(\mathbf{R}^n)}.$$

Proof. By the Plancherel's theorem :

$$\begin{aligned}\|K * f\|_{\mathrm{L}^2(\mathbf{R}^n)} &= \|(K * f)^\wedge\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)} \\ &= \|\hat{K} \cdot \hat{f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)} \\ &\leqslant \|\hat{K}\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} \|\hat{f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)} \\ &= \|\hat{K}\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} \|f\|_{\mathrm{L}^2(\mathbf{R}^n)}.\end{aligned}$$

□

Lemma 5.4. *Let K be a kernel of a Calderón-Zygmund operator and let Q be a compact cube. Suppose $f \in \mathrm{L}^1(\mathbf{R}^n)$ has mean zero and is such that* $\mathrm{supp} f \subset Q$*.*

$$\|K * f\|_{\mathrm{L}^1(\mathbf{R}^n \backslash Q^*)} \lesssim \|f\|_{\mathrm{L}^1(\mathbf{R}^n)}$$

where Q^* is the cube concentric to Q but with $2\sqrt{n}$ times the side-length.

Proof. Assume that f is not zero almost everywhere, otherwise the result is trivial. Hence we can assume that $|Q| > 0$. We fix $x \in \mathbf{R}^n \backslash Q^*$. Then for all $y \in \mathrm{supp} f$, $|x - y| \geqslant |Q^*| > 0$ so

$$K * f(x) = \int_{\mathbf{R}^n} K(x - y) f(y) \mathrm{d}y$$

because a kernel of a Calderón-Zygmund operator coincides with a locally integrable function on $\mathbf{R}^n \backslash \{0\}$. Let c be the center of the cube Q. Since $f \in \mathrm{L}^1(\mathbf{R}^n)$ has mean zero, we can write

$$K * f(x) = \int_{\mathbf{R}^n} (K(x - y) - K(x - c)) f(y) \mathrm{d}y. \tag{5.5}$$

Note that if $y \in Q$ we have

$$|x - c| \geqslant \sqrt{n}\, l(Q) \geqslant 2|y - c|. \tag{5.6}$$

In fact the minimum distance between x and c is $\sqrt{n}\, l(Q)$, and the maximal distance between y and c is, by the theorem of Pythagore, $\frac{\sqrt{n}}{2} l(Q)$ (see the figure below).

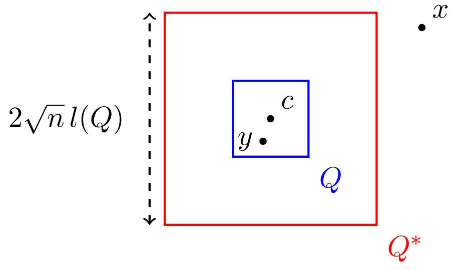

Figure 3: Q and Q^*

Let's conclude the proof. By the Fubini-Tonelli's theorem, using (5.5) we have

$$\begin{aligned}\|K * f\|_{\mathrm{L}^1(\mathbf{R}^n\backslash Q^*)} &= \int_{\mathbf{R}^n\backslash Q^*}\left|\int_{\mathbf{R}^n}(K(x-y)-K(x-c))f(y)\mathrm{d}y\right|\mathrm{d}x \\ &\leqslant \int_{\mathbf{R}^n\backslash Q^*}\int_{\mathbf{R}^n}|K(x-y)-K(x-c)||f(y)|\mathrm{d}y\mathrm{d}x \\ &= \int_{\mathbf{R}^n\backslash Q^*}\int_{Q}|K(x-y)-K(x-c)||f(y)|\mathrm{d}y\mathrm{d}x.\end{aligned}$$

By using again the Fubini-Tonelli's theorem we obtain

$$\|K * f\|_{\mathrm{L}^1(\mathbf{R}^n\backslash Q^*)} \leqslant \int_Q\left(\int_{\mathbf{R}^n\backslash Q^*}|K(x-y)-K(x-c)|\mathrm{d}x\right)|f(y)|\mathrm{d}y.$$

By (5.6), we know that $\mathbf{R}^n\backslash Q^* \subset \{x \in \mathbf{R}^n : |x-c| \geqslant 2|y-c|\}$ if $y \in Q$. So it comes

$$\begin{aligned}\|K * f\|_{\mathrm{L}^1(\mathbf{R}^n\backslash Q^*)} &\leqslant \int_Q\left(\int_{|x-c|\geqslant 2|y-c|}|K(x-y)-K(x-c)|\mathrm{d}x\right)|f(y)|\mathrm{d}y \\ &= \int_Q\left(\int_{|x-c|\geqslant 2|y-c|}|K((x-c)-(y-c))-K(x-c)|\mathrm{d}x\right)|f(y)|\mathrm{d}y \\ &= \int_Q\left(\int_{|x|\geqslant 2|y|}|K((x-y))-K(x)|\mathrm{d}x\right)|f(y)|\mathrm{d}y\end{aligned}$$

and by the assumption (5.3) in the definition of Calderón-Zygmund operator we obtain the desired result. □

Let K be a kernel of a Calderón-Zygmund operator. Assume that a function $f \in \mathrm{L}^1(\mathbf{R}^n)$ is given and we are able to break it as $f = g + b$, as in the introduction. Since g is bounded, the lemma 5.3 can gives us a good bound for $K * g$. **Heuristically**, for the bad part we have

$$\|K * b\|_{\mathrm{L}^1(\mathbf{R}^n)} = \|K * b\|_{\mathrm{L}^1(Q^*)} + \|K * b\|_{\mathrm{L}^1(\mathbf{R}^n\backslash Q^*)} \lesssim |Q^*| + \|K * f\|_{\mathrm{L}^1(\mathbf{R}^n\backslash Q^*)}$$

and then by the lemma 5.4 and the fact that Q is small, this bounded should be small. In fact it's not that simple, the support of b will be contained in an union of cubes, but it's the main idea. Then some interpolation arguments will conclude the proof in $\mathrm{L}^p(\mathbf{R}^n)$ for $1 < p < \infty$.
Let's move on the proof of the decomposition. The proof is based on a fundamental lemma : the Calderón-Zygmund covering lemma, which gives us a decomposition of $\mathbf{R}^n$ adapted to an integrable function.

Lemma 5.5 (Calderón-Zygmund covering lemma). *Let $f \in \mathrm{L}^1(\mathbf{R}^n)$ and $\lambda > 0$. Then there exists a collection of disjoint dyadic cubes $\mathcal{Q} \subset \mathcal{D}^n$ such that :*

(*i*) *for almost every* $x \notin \bigcup_{Q\in\mathcal{Q}} Q$, $|f(x)| \leqslant \lambda$;

(*ii*)

$$\left|\bigcup_{Q\in\mathcal{Q}} Q\right| \leqslant \frac{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}}{\lambda};$$

(*iii*) *for all* $Q \in \mathcal{Q}$

$$\lambda < \fint_Q |f| \leqslant 2^n\lambda.$$

Proof. We are going the weak-type $(1,1)$ of the dyadic maximal function, theorem 2.14, and its corollary (proposition 3.4). As in the proof of theorem 2.14, we form for $k \in \mathbf{Z}$ the sets

$$\Omega_k := \{x \in \mathbf{R}^n : E_k f(x) > \lambda \text{ and } E_j f(x) \leqslant \lambda \text{ if } j > k\}.$$

and decompose each into disjoint dyadic cubes contained in $\mathcal{D}_k$. All of these cubes form the family $\mathcal{Q}$.

(*i*) Let $x \notin \cup_{Q\in\mathcal{Q}} Q$. Then for every $k \in \mathbf{Z}$, $x \notin \Omega_k$ so $E_k f(x) \leqslant \lambda$. Then by the proposition 3.4, we obtain the results by taking the limit as $k \to \infty$.

(*ii*) The second point is just the weak $(1,1)$ inequality of the theorem 2.14.

(*iii*) Let $Q \in \mathcal{Q}$. According to the definition of the sets Ω_k, the average of f over the cubes Q is greater than λ :

$$\fint_Q f > \lambda.$$

Let Q^* be the cube with the same center as Q but such that $l(Q^*) = 2l(Q)$. So by the definition of the sets Ω_k the average of f over Q^* is at most λ. Hence

$$\fint_Q f \leqslant \frac{|Q^*|}{|Q|}\fint_{Q^*} f \leqslant 2^n\lambda.$$

□

We can finally prove the desired decomposition result.

Corollary 5.6 (Calderón-Zygmund decomposition). *Let* $f \in \mathrm{L}^1(\mathbf{R}^n)$ *and* $\lambda > 0$. *Then there exists a collection* $\mathcal{Q} \subset \mathcal{D}^n$ *of disjoint dyadic cubes and some functions* $g, b_Q \in \mathrm{L}^1(\mathbf{R}^n)$ *(for* $Q \in \mathcal{Q}$*) such that*

$$f = g + b, \quad \text{for} \quad b := \sum_{Q\in\mathcal{Q}} b_Q.$$

where $\|g\|_{\mathrm{L}^1(\mathbf{R}^n)} \leqslant \|f\|_{\mathrm{L}^1(\mathbf{R}^n)}$ *and for all* $Q \in \mathcal{Q}$, $\|b_Q\|_{\mathrm{L}^1(\mathbf{R}^n)} \leqslant 2\|f\|_{\mathrm{L}^1(Q)}$. *Furthermore :*

(*i*) *for almost every* $x \in \mathbf{R}^n$,

$$|g(x)| \leqslant 2^n\lambda;$$

(*ii*) *for all* $Q \in \mathcal{Q}$

$$\operatorname{supp}(b_Q) \subset Q \quad et \quad \int_Q b_Q = 0;$$

(*iii*)

$$\left|\bigcup_{Q\in\mathcal{Q}} Q\right| \leqslant \frac{1}{\lambda}\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}.$$

Proof. Let $\mathcal{Q}$ be the collection of cubes given by the lemma 5.5. Let's define the bad functions for all $Q \in \mathcal{Q}$:

$$b_Q := \left(f - \fint_Q f\right)\chi_Q,$$

and let the good function be

$$g := f - \sum_{Q\in\mathcal{Q}} b_Q.$$

These first results are clear :

- $\mathrm{supp}(b_Q) \subset Q$;
- $\int_Q b_Q = 0$;
- $\|b_Q\|_{\mathrm{L}^1(\mathbf{R}^n)} \leqslant 2\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}$.

On the one hand, for all $x \notin \cup_{Q\in\mathcal{Q}} Q$, $g(x) = f(x) \leqslant \lambda \leqslant 2^n\lambda$. On the other hand, if $x \in Q$ for some $Q \in \mathcal{Q}$,

$$|g(x)| = \left|\fint_Q f\right| \leqslant \fint_Q |f| \leqslant 2^n\lambda.$$

So for almost every $x \in \mathbf{R}^n$, $|g(x)| \leqslant 2^n\lambda$, we got the first point. We just have to prove that $\|g\|_{\mathrm{L}^1(\mathbf{R}^n)} \leqslant \|f\|_{\mathrm{L}^1(\mathbf{R}^n)}$.

$$\|g\|_{\mathrm{L}^1(\mathbf{R}^n)} = \int_{\mathbf{R}^n\setminus\bigcup_{Q\in\mathcal{Q}} Q} |g| + \int_{\bigcup_{Q\in\mathcal{Q}} Q} |g|,$$

and since the cubes are disjoint

$$\|g\|_{\mathrm{L}^1(\mathbf{R}^n)} \leqslant \int_{\mathbf{R}^n\setminus\bigcup_{Q\in\mathcal{Q}} Q} |f| + \sum_{Q\in\mathcal{Q}} \int_Q |g|$$

and $\int_Q |g| \leqslant \int_Q |f|$, which concludes the proof. □

5.3. **Proof**

We can now prove the Calderón-Zygmund theorem. Let K be a tempered distribution as in the theorem 5.1. We note T the convolution operator associated to the kernel K :

$$Tf := K * f,$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$.

5.3.1. **Step 1 : Reduction of the problem.**

In fact it suffices to show that T is weak $(1,1)$. We use interpolation and duality. Assume that T is weak $(1,1)$. By the lemma 5.3, we know that

$$\|Tf\|_{\mathrm{L}^2(\mathbf{R}^n)} \lesssim \|\hat{K}\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} \|f\|_{\mathrm{L}^2(\mathbf{R}^n)}$$

holds whenever $f \in \mathcal{S}(\mathbf{R}^n)$. Since thee space $\mathcal{S}(\mathbf{R}^n)$ is dense in $\mathrm{L}^2(\mathbf{R}^n)$, this result can be extended to all $f \in \mathrm{L}^2(\mathbf{R}^n)$. So T is strong (p,p). By using the theorem 2.9, this result of interpolation asserts that T is strong (p,p) for $1 < p \leqslant 2$. Now let $2 < p < \infty$, and denote p' its conjugate index. For all $f, g \in \mathcal{S}(\mathbf{R}^n)$,

$$\langle Tf, g\rangle = \langle f, T^*g\rangle$$

where $\langle \cdot, \cdot \rangle$ denote the inner product in $\mathrm{L}^2(\mathbf{R}^n)$, and T^* the adjoint operator of T. It's clear that the adjoint operator T^* has kernel $K^* = K(-\cdot)$ which also satisfies (5.2) and (5.3), so T^* is also a Calderón-Zygmund operator. Since $1 < p' < 2$, T^* is srong (p', p'). Let $f \in \mathrm{L}^p(\mathbf{R}^n)$. By using the extremal equality of Holder's inequality we have

$$\begin{aligned} \|Tf\|_{\mathrm{L}^p(\mathbf{R}^n)} &= \sup \left\{ \left| \int_{\mathbf{R}^n} Tf \cdot g \right| : g \in \mathrm{L}^{p'}(\mathbf{R}^n), \|g\|_{\mathrm{L}^{p'}(\mathbf{R}^n)} \leqslant 1 \right\} \\ &= \sup \left\{ |\langle f, T^* g \rangle| : g \in \mathrm{L}^{p'}(\mathbf{R}^n), \|g\|_{\mathrm{L}^{p'}(\mathbf{R}^n)} \leqslant 1 \right\} \\ &\leqslant \|f\|_{\mathrm{L}^p(\mathbf{R}^n)} \sup \left\{ \|T^* g\|_{\mathrm{L}^{p'}(\mathbf{R}^n)} : g \in \mathrm{L}^{p'}(\mathbf{R}^n), \|g\|_{\mathrm{L}^{p'}(\mathbf{R}^n)} \leqslant 1 \right\} \end{aligned}$$

and because T^* is strong (p', p')

$$\|Tf\|_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim \|f\|_{\mathrm{L}^p(\mathbf{R}^n)}.$$

5.3.2. **Step 2 : Proof of the weak $(1,1)$ bound.**

Now we have to prove the weak-type $(1,1)$ of T. Let $f \in \mathrm{L}^1(\mathbf{R}^n)$ and $\lambda > 0$. We use the Calderón-Zygmund decomposition on the function $f = g + b$, where g, b are the good and bad parts of f. We write

$$|\{x \in \mathbf{R}^n : |Tf(x)| > \lambda\}| \leqslant |\{x \in \mathbf{R}^n : |Tg(x)| > \lambda/2\}| + |\{x \in \mathbf{R}^n : |Tb(x)| > \lambda/2\}|. \quad (5.7)$$

Since g is bounded almost everywhere, the first term is easly bounded with the Markov's inequality :

$$\begin{aligned} |\{x \in \mathbf{R}^n : |Tg(x)| > \lambda/2\}| &= |\{x \in \mathbf{R}^n : |Tg(x)|^2 > \lambda^2/4\}| \\ &\leqslant \frac{4}{\lambda^2} \|Tg\|_{\mathrm{L}^2(\mathbf{R}^n)}. \end{aligned}$$

Then by the lemma 5.3 and the (i) of the corollary 5.6 we obtain

$$|\{x \in \mathbf{R}^n : |Tg(x)| > \lambda/2\}| \lesssim \frac{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}}{\lambda}. \quad (5.8)$$

For the second term we have to use the collection of cubes $\mathcal{Q} \subset \mathcal{D}^n$ given by the decomposition of Calderón-Zygmund. We have

$$|\{x \in \mathbf{R}^n : |Tb(x)| > \lambda/2\}| \leqslant \left| \bigcup_{Q \in \mathcal{Q}} Q^* \right| + \left| \left\{ x \notin \bigcup_{Q \in \mathcal{Q}} Q^* : |Tb(x)| > \lambda/2 \right\} \right|.$$

By the third point of the corollary 5.6, we have

$$\left| \bigcup_{Q \in \mathcal{Q}} Q^* \right| \lesssim \frac{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}}{\lambda}$$

where the constant depends on n. And by the Markov's inequality

$$|\{x \in \mathbf{R}^n \backslash \bigcup_{Q \in \mathcal{Q}} Q : |Tb(x)| > \lambda/2\}| \leqslant \frac{2}{\lambda} \|Tb\|_{\mathrm{L}^1(\mathbf{R}^n \backslash \cup_{Q \in \mathcal{Q}} Q)}.$$

By using the lemma 5.4, the corollary 5.6 and since the cubes are disjoint

$$\|Tb\|_{\mathrm{L}^1(\mathbf{R}^n \backslash \cup_{Q \in \mathcal{Q}} Q)} = \| \sum_{Q \in \mathcal{Q}} Tb_Q \|_{\mathrm{L}^1(\mathbf{R}^n \backslash \cup_{Q \in \mathcal{Q}} Q)} \lesssim \sum_{Q \in \mathcal{Q}} \|b_Q\|_{\mathrm{L}^1(Q)} \lesssim \|f\|_{\mathrm{L}^1(\mathbf{R}^n)}.$$

So

$$|\{x \in \mathbf{R}^n : |Tb(x)| > \lambda/2\}| \lesssim \frac{\|f\|_{\mathrm{L}^1(\mathbf{R}^n)}}{\lambda}. \quad (5.9)$$

By the equalities (5.7), (5.8) and (5.9) we finally obtain the weak $(1,1)$ bound.

6. The Hörmander–Mikhlin Multiplier Theorem

We are going to present a theorem with many application, especially in PDEs. Its a direct application of the Calderón-Zygmund theory.

6.1. Littlewood-Paley decomposition

Here we are going to decompose a function f into a sum of functions with localized frequencies. Fix $\eta \in \mathcal{C}^\infty(\mathbf{R}^n)$ satisfying

$$\operatorname{supp}(\eta) \subset [-2,2] \quad \text{and} \quad \eta(r) = 1 \text{ if } |r| \leqslant 1.$$

We define $\beta, \beta_j \in \mathcal{C}_c^\infty(\mathbf{R})$, for $j \in \mathbf{Z}$ by

$$\beta : \left| \begin{array}{ccc} \mathbf{R} & \longrightarrow & \mathbf{R} \\ r & \longmapsto & \eta(r) - \eta(2r) \end{array} \right.$$

and

$$\beta_j := \beta(2^{-j}\cdot).$$

Hence we have

$$\operatorname{supp}(\beta) \subset [-2,-1/2] \cup [1/2,2] \quad \text{and} \quad \operatorname{supp}(\beta_j) \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}]$$

for all $j \in \mathbf{Z}$, and for all $r \in \mathbf{R}\backslash\{0\}$

$$\sum_{j\in\mathbf{Z}} \beta_j(r) = 1. \tag{6.1}$$

We identify the functions β_j on $\widehat{\mathbf{R}}$ and the radial functions $\beta_j(|\cdot|)$ on $\widehat{\mathbf{R}}^n$. Then we consider the **Littlewood-Paley functions** $\beta_j \in \mathcal{C}^\infty(\widehat{\mathbf{R}}^n)$. So each function β_j is supported in the **dyadic annulus**

$$A_j := \{\xi \in \widehat{\mathbf{R}}^n : 2^{j-1} \leqslant |\xi| < 2^{j+1}\}.$$

and the collection of all such functions forms a smooth partition of unity of $\widehat{\mathbf{R}}^n$ adapted to the covering $\{A_j\}_{j\in\mathbf{Z}}$. This partition of unity is called the **smooth Littlewood–Paley decomposition**.

6.2. The Hörmander–Mikhlin multiplier theorem

This theorem is an application of the Littlewood-Paley decomposition as well as an application of the Calderón-Zygmund theory. We defined the notion of Fourier multipliers earlier, see the definition 4.1.

Theorem 6.1 (Hörmander-Mikhlin multipliers). *Suppose $m \in \mathrm{L}^\infty(\widehat{\mathbf{R}}^n)$ is smooth away from zero and is such that*

$$\left|\partial_\xi^\alpha m(\xi)\right| \lesssim_\alpha |\xi|^{-|\alpha|}$$

for all $\xi \in \widehat{\mathbf{R}}^n\backslash\{0\}$ and for all multi-index $\alpha \in \mathbf{N}^n$. Then the Fourier multiplier operator T_m extends to a bounded operator on $L^p(\mathbf{R}^n)$, for $1 < p < \infty$.

By abuse of language, such a function is called a **Hörmander-Mikhlin multipliers**.

Examples 6.2. (*i*) The Littlewood-Paley functions β_j for $j \in \mathbf{Z}$ are Hörmander-Mikhlin multipliers.

(*ii*) For $N \geqslant 2$ et $1 \leqslant j \leqslant N$, we consider the multiplier

$$m_j(\xi) := -i\frac{\xi_j}{|\xi|}, \quad \forall \xi \in \widehat{\mathbf{R}}\backslash\{0\}.$$

The operators $R_j := T_{m_j}$ are called the **Riesz transform**. It's clear that they are some Hörmander-Mikhlin multipliers. So by the theorem 6.1, R_j extends to all $\mathrm{L}^p(\mathbf{R}^n)$, $1 < p < \infty$, and for all $f \in \mathrm{L}^p(\mathbf{R}^n)$

$$\||R_j f\|_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim \|f\|_{\mathrm{L}^p(\mathbf{R}^n)}.$$

Furthermore, for all $1 \leqslant j,k \leqslant n$ and for all $f \in \mathcal{S}(\mathbf{R}^n)$ we have

$$\partial^2_{x_j,x_k} f = -R_j R_k \Delta f \tag{6.2}$$

(it suffices to determinate the Fourier transform of both term).

The Calderón-Zygmund has a large range of applications to PDE, the equality (6.2) gives us an example.

Corollary 6.3. *Let $1 < p < \infty$ and $1 \leqslant j,k \leqslant n$. Then for all $f \in \mathcal{S}(\mathbf{R}^n)$*

$$\|\partial^2_{x_j,x_k} f\|_{\mathrm{L}^p(\mathbf{R}^n)} \lesssim_{p,n} \|\Delta f\|_{\mathrm{L}^p(\mathbf{R}^n)}.$$

Proof. It suffices to use the equality (6.2) and the theorem 6.1. □

6.3. **Proof**

First, let's explain the main ideas of the proof in a **heuristic way**. Let m be a Hörmander-Mikhlin multipliers, and $K := \check{m}$ the kernel associated to the multiplier T_m. We know that

$$T_m f = K * f, \quad \text{for all } f \in \mathcal{S}(\mathbf{R}^n). \tag{6.3}$$

We want to apply the Calderón-Zygmund on K. Assume we know that K coincides with a function away from the origin. Then if $m \in \mathrm{L}^1(\widehat{\mathbf{R}}^n)$, K is a function defined by

$$K : x \in \mathbf{R}^n \longmapsto \int_{\widehat{\mathbf{R}}^n} m(\xi) e^{2i\pi x\cdot\xi} \mathrm{d}\xi.$$

However, we don't know if m is integrable. Then we use the smooth Littlwood-Paley decomposition and we decompose m as

$$m = \sum_{j\in\mathbf{Z}} m_j, \quad \text{where } m_j := m\beta_j. \tag{6.4}$$

For all $j \in \mathbf{Z}$, $\mathrm{supp}(m_j) \subset A_j$ so $m_j \in \mathrm{L}^1(\widehat{\mathbf{R}}^n)$. Now we can work on each pieces K_j of K, and at the end of the day we get back K via a limiting procedure. Let's see the details.

Proof. We assumed that $m \in \mathrm{L}^\infty(\widehat{\mathbf{R}}^n)$, since $K = \check{m}$, the condition (5.2) holds. We decompose m as in (6.4),

$$T_m f = \sum_{j\in\mathbf{Z}} T_{m_j} f, \quad \forall f \in \mathrm{L}^2(\mathbf{R}^n),$$

the convergence holds in $\mathrm{L}^2(\mathbf{R}^n)$. Let's check that point. Let $f \in \mathrm{L}^2(\mathbf{R}^n)$ and $N \in \mathbf{N}$. Then by the Plancherel's theorem

$$\begin{aligned}\|T_m f - \sum_{j=-N}^{N} T_{m_j} f\|_{\mathrm{L}^2(\mathbf{R}^n)} &= \| \left(\hat{f} - \sum_{j=-N}^{N} \beta_j \hat{f}\right) m\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)} \\ &\leqslant \|m\|_{\mathrm{L}^\infty(\widehat{\mathbf{R}}^n)} \|\hat{f} - \sum_{j=-N}^{N} \beta_j \hat{f}\|_{\mathrm{L}^2(\widehat{\mathbf{R}}^n)}.\end{aligned}$$

Then we use the dominated convergence theorem and the equality (6.1) to conclude. Since m and the β_j are Hörmander-Mikhlin multipliers, the product m_j is also Hörmander-Mikhlin multipliers (the details are left to the reader), so for all $\xi \in \widehat{\mathbf{R}}^n \backslash \{0\}$ and all $\alpha \in \mathbf{N}^n$ $\left|\partial_\xi^\alpha m_j(\xi)\right| \lesssim_\alpha |\xi|^{-|\alpha|}$. Furthermore $\operatorname{supp}(m_j) \subset A_j$, so by integrating we have

$$\begin{aligned} \|\partial_\xi^\alpha m_j(\xi)\|_{\mathrm{L}^1(\widehat{\mathbf{R}}^n)} &\lesssim_\alpha \int_{A_j} |\xi|^{-|\alpha|} \mathrm{d}\xi \\ &\lesssim_\alpha 2^\alpha 2^{-j\alpha} |A_j| \end{aligned}$$

and we finally obtain

$$\|\partial_\xi^\alpha m_j(\xi)\|_{\mathrm{L}^1(\widehat{\mathbf{R}}^n)} \lesssim_\alpha 2^{-j\alpha} 2^{jn}. \tag{6.5}$$

Taking $\alpha = 0$, we see $m_j \in \mathrm{L}^1(\widehat{\mathbf{R}}^n)$, and so we can define K_j the kernel of the multiplier T_{m_j} by

$$K_j(x) = \int_{\widehat{\mathbf{R}}^n} m_j(\xi) e^{2i\pi x \cdot \xi} \mathrm{d}\xi, \quad \forall x \in \mathbf{R}^n,$$

for $x \in \mathbf{R}^n$.

Now we have to realise the kernel K is the distributional limit of the series $\sum_{j\in\mathbf{Z}} K_j$, and to show this limits agrees with a function away from zero. For $x \in \mathbf{R}^n \backslash \{0\}$, we define the vector fields

$$\langle x, \partial_\xi \rangle := \sum_{k=1}^n x_k \partial_{\xi_k}.$$

In view of the differential identity

$$\frac{1}{2i\pi} \frac{\langle x, \partial_\xi \rangle}{|x|^2} e^{2i\pi x\cdot\xi} = e^{2i\pi x \cdot \xi},$$

by repeating integration-by-parts, and since $\operatorname{supp}(m_j)$ is compact, we obtain for all $N \in \mathbf{N}$,

$$K_j(x) = \left(\frac{-1}{2i\pi |x|^2} \right)^N \int_{\widehat{\mathbf{R}}^n} e^{2i\pi x\cdot\xi} \langle x, \partial_\xi \rangle^N m_j(\xi) \mathrm{d}\xi. \tag{6.6}$$

Combining the equality (6.6) with the inequaliy (6.5), we obtain for all $N \in \mathbf{N}$,

$$|K_j(x)| \lesssim_N 2^{jn} (1 + 2^j |x|)^{-N}. \tag{6.7}$$

So the serie $\sum_{j\in\mathbf{Z}} K_j$ converges pointwise and absolutely on $\mathbf{R}^n \backslash \{0\}$ to a function $\tilde{K}$. By taking $N = n+1$ in (6.7), we have

$$\sum_{j\in\mathbf{Z}} |K_j(x)| \lesssim \sum_{\substack{j\in\mathbf{Z} \\ 2^j|x| \leqslant 1}} 2^{jn} + \sum_{\substack{j\in\mathbf{Z} \\ 2^j|x|>1}} 2^{-j} |x|^{-(n+1)}.$$

Then by evaluating these geometric series, we see that $\tilde{K}$ satisfies $|\tilde{K}(x)| \lesssim |x|^{-n}$. The pointwise limit (6.4) also holds in the sense of distributions. Then

$$K = \sum_{j\in\mathbf{Z}} K_j$$

where the convergence is in the sense of distributions. Then K agrees with $\tilde{K}$ away from zero, so we drop the tilde notation.

We finally have to show that K satisfies the Hörmander condition (5.3). We will show that K satisfies the Hörmander stronger condition, see the proposition 5.2. We have

$$\partial_{x_j} K_j(x) = 2i\pi \int_{\widehat{\mathbf{R}}^n} \xi_j m_j(\xi) e^{2i\pi x\cdot\xi} \mathrm{d}\xi.$$

As above we can show that for all $N \in \mathbf{N}$

$$|\partial_x K_j(x)| \lesssim_N 2^{j(n+1)}(1 + 2^j|x|)^{-N}$$

and for all $x \in \mathbf{R}^n \backslash \{0\}$, by taking $N = n + 2$,

$$|\nabla K(x)| \lesssim |x|^{-n-1}.$$

Then K satisfies the condition (5.3), so by applying the theorem 5.1 we conclude the proof. □

References

[1] Samuel Chan-Ashing, Bastien Lecluse, *Principe d'incertitude et théorème de Logvinenko-Sereda*, Lecture dirigée, ENS Rennes, 2021.

[2] Yutae Choi, Younhwoo Koh, Jungjin Lee, *Note on Nikodym maximal in the variable Lebesgue spaces*, 2016.

[3] Pierre Colmez, *Éléments d'analyse et d'algèbre (et de théorie des nombres)*, 2009.

[4] Javier Duoandikoetxea, *Fourier Analysis*, Graduate Studies in Mathematics 29, American Mathematical Society, Providence, RI, 2001.

[5] Chengchun Hao, *Introduction to Harmonic Analysis*, Lectures notes, Institute of Mathematics, AMSS, CAS, 2016.

[6] Jonathan Hickman, *Fourier and Harmonic Analysis*, University of Edinburgh, SM-STC 2021-22.

[7] Wihelm Schlag, *A geometric proof of the circular maximal theorem.*

[8] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993.

[9] Elias M. Stein, Guido Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton, University Press, Princeton, N.J., 1971.

[10] Terence Tao, *The Bochner–Riesz conjecture implies the restriction conjecture*, Duke Math. 96(2), 363–375 (1999).